

Chapter 3 Fourier Series

Def. Periodic function

A function $f(t) \in R$, if $f(t+2p) = f(t)$, for any t , then f is a periodic function with the period $2p$.

Ex. $\sin(\omega t) = \sin[\omega(t + \frac{2\pi}{\omega})]$ is a periodic function with the period $\frac{2\pi}{\omega}$.

Ex. $f(t) = c$ is periodic function with the period $2p$, $p \in R$.

$1, \cos \frac{\pi t}{p}, \sin \frac{\pi t}{p}, \dots, \sin \frac{n\pi t}{p}$ are linearly independent on R , $t \in (0, 2p)$

Then a real and continuous function $f(t)$, $t \in (0, 2p)$, can be

approximated as

$$f(t) = a_0 + a_1 \cos \frac{\pi t}{p} + b_1 \sin \frac{\pi t}{p} + \dots + a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p}$$

such that the least square

$$E = \int_0^{2p} \{f(t) - a_0 - a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} - \dots - a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p}\}^2 dx$$

to be minimum.

The conditions of E to be minimum are $\frac{\partial E}{\partial a_i} = 0$ and $\frac{\partial E}{\partial b_i} = 0$:

$$\frac{\partial E}{\partial a_0} = \int_0^{2p} \{f(t) - a_0 - a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} - \dots - a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p}\} dx = 0$$

$$\frac{\partial E}{\partial a_i} = \int_0^{2p} \{f(t) - a_0 - a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} - \dots - a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p}\} \cos \frac{i\pi t}{p} dx = 0$$

$$\frac{\partial E}{\partial b_i} = \int_0^{2p} \{f(t) - a_0 - a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} - \dots - a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p}\} \sin \frac{i\pi t}{p} dx = 0$$

$$\int_0^{2p} \cos \frac{i\pi t}{p} dt = \frac{p}{i\pi} \sin \frac{i\pi t}{p} \Big|_0^{2p} = 0, \quad \int_0^{2p} \sin \frac{i\pi t}{p} dt = -\frac{p}{i\pi} \cos \frac{i\pi t}{p} \Big|_0^{2p} = 0$$

$$\begin{aligned} \int_0^{2p} \cos \frac{i\pi t}{p} \cos \frac{j\pi t}{p} dt &= \frac{1}{2} \int_0^{2p} [\cos \frac{(i+j)\pi t}{p} + \cos \frac{(i-j)\pi t}{p}] dt \\ &= \frac{1}{2} \int_0^{2p} \cos \frac{(i-j)\pi t}{p} dt = \begin{cases} 0, & i \neq j \\ p, & i = j \end{cases} \end{aligned}$$

Similarly,

$$\int_0^{2p} \sin \frac{i\pi t}{p} \sin \frac{j\pi t}{p} dt = \begin{cases} 0, & i \neq j \\ p, & i = j \end{cases}$$

$$a_0 = \frac{1}{2p} \int_0^{2p} f(t) dt, \quad a_i = \frac{1}{p} \int_0^{2p} f(t) \cos \frac{i\pi t}{p} dt, \quad b_i = \frac{1}{p} \int_0^{2p} f(t) \sin \frac{i\pi t}{p} dt$$

Ex. Expand the function

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ \sin t & 0 < t < \pi \end{cases} \text{ into the Fourier series, } p = \pi$$

$$f(t) = a_0 + a_1 \cos t + b_1 \sin t + \dots a_n \cos nt + b_n \sin nt$$

where

$$a_0 = \frac{1}{2\pi} \left[\int_{-\pi}^0 f(t) dt + \int_0^{\pi} f(t) dt \right] = \frac{1}{2\pi} \int_0^{\pi} \sin t dt = -\frac{\cos t \Big|_0^{\pi}}{2\pi} = \frac{1}{\pi}$$

$$b_i = \frac{1}{\pi} \int_0^{\pi} f(t) \sin(it) dt = \frac{1}{\pi} \int_0^{\pi} \sin t \sin it dt = \frac{1}{\pi} \int_0^{\pi} [\cos(i-1)t - \cos(i+1)t] dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \cos(i-1)t dt = \begin{cases} 1/2 & i=1 \\ 0 & i \neq 1 \end{cases}$$

$$a_i = \frac{1}{\pi} \int_0^{\pi} f(t) \cos(it) dt = \frac{1}{2\pi} \int_0^{\pi} \sin t \cos it dt = \frac{1}{2\pi} \int_0^{\pi} [\sin(1-i)t + \sin(i+1)t] dt$$

$$a_1 = 0$$

$$a_i = \frac{1}{2\pi} \left\{ -\frac{\cos(1-i)t}{1-i} - \frac{\cos(1+i)t}{1+i} \right\} \Big|_0^{\pi} = \frac{1}{2\pi} \left\{ \frac{1 - \cos(1-i)\pi}{1-i} + \frac{1 - \cos(1+i)\pi}{1+i} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{1 + \cos i\pi}{1-i} + \frac{1 + \cos i\pi}{1+i} \right\} = \frac{1 + \cos i\pi}{\pi(1-i^2)} = \frac{1 + (-1)^i}{\pi(1-i^2)}$$

$$f(t) = \frac{1}{\pi} + \frac{\sin t}{2} - \frac{2}{\pi} \left(\frac{\cos 2t}{3} + \frac{\cos 4t}{15} + \frac{\cos 6t}{35} + \dots \right)$$

Hw. Find the Fourier series of the following functions

a) $f(t) = t, \quad -\pi < t < \pi$, b) $f(t) = t^2, \quad -\pi < t < \pi$

c) $f(t) = \begin{cases} 0 & -\pi \leq x < 0 \\ x^2 & 0 < x < \pi \end{cases}$

§ If $f(t) = f(-t)$, then f is symmetric

If $f(t) = -f(-t)$, then f is skew-symmetric

If f is symmetric, then

$$f(t) = a_0 + a_1 \cos \frac{\pi t}{p} + b_1 \sin \frac{\pi t}{p} + \dots a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p}$$

$$f(t) = f(-t) = a_0 + a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} + \dots a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p}$$

implies $b_i = 0$ and

$$\begin{aligned} a_0 &= \frac{1}{2p} \int_{-p}^p f(t) dt = \frac{1}{2p} \left[\int_{-p}^0 f(t) dt + \int_0^p f(t) dt \right] = \frac{1}{2p} \left[\int_p^0 f(-t) d(-t) + \int_0^p f(t) dt \right] \\ &= \frac{1}{2p} \left[- \int_p^0 f(t) dt + \int_0^p f(t) dt \right] = \frac{1}{p} \int_0^p f(t) dt \\ a_i &= \frac{1}{p} \int_{-p}^p f(t) \cos \frac{i\pi t}{p} dt = \frac{1}{p} \left[\int_{-p}^0 f(t) \cos \frac{i\pi t}{p} dt + \int_0^p f(t) \cos \frac{i\pi t}{p} dt \right] \\ &= \frac{1}{p} \left[\int_p^0 f(-t) \cos \left(-\frac{i\pi t}{p} \right) d(-t) + \int_0^p f(t) \cos \frac{i\pi t}{p} dt \right] \\ &= \frac{1}{p} \left[- \int_p^0 f(t) \cos \left(\frac{i\pi t}{p} \right) dt + \int_0^p f(t) \cos \frac{i\pi t}{p} dt \right] = \frac{2}{p} \int_0^p f(t) \cos \frac{i\pi t}{p} dt . \end{aligned}$$

If f is skew-symmetric, then

$$\begin{aligned} f(t) &= a_0 + a_1 \cos \frac{\pi t}{p} + b_1 \sin \frac{\pi t}{p} + \dots a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \\ -f(t) &= f(-t) = a_0 + a_1 \cos \frac{\pi t}{p} - b_1 \sin \frac{\pi t}{p} + \dots a_n \cos \frac{n\pi t}{p} - b_n \sin \frac{n\pi t}{p} \end{aligned}$$

implies $a_i = 0$ and

$$\begin{aligned} b_i &= \frac{1}{p} \int_{-p}^p f(t) \sin \frac{i\pi t}{p} dt = \frac{1}{p} \left[\int_{-p}^0 f(t) \sin \frac{i\pi t}{p} dt + \int_0^p f(t) \sin \frac{i\pi t}{p} dt \right] \\ &= \frac{1}{p} \left[\int_p^0 f(-t) \sin \left(-\frac{i\pi t}{p} \right) d(-t) + \int_0^p f(t) \sin \frac{i\pi t}{p} dt \right] = \frac{1}{p} \left[- \int_p^0 f(t) \sin \left(\frac{i\pi t}{p} \right) dt + \int_0^p f(t) \sin \frac{i\pi t}{p} dt \right] \\ &= \frac{2}{p} \int_0^p f(t) \sin \frac{i\pi t}{p} dt \end{aligned}$$

Ex. $f(t) = 4 - t^2 \quad -2 < t < 2$

$$f(t) = a_0 + a_1 \cos \left(\frac{\pi t}{2} \right) + a_2 (\pi t) + a_3 \left(\frac{3\pi t}{2} \right) + a_4 (2\pi t) + \dots$$

$$a_0 = \frac{1}{2} \int_0^2 (4 - t^2) dt = \frac{1}{2} \left[8 - \frac{8}{3} \right] = \frac{8}{3}$$

$$a_i = \frac{2}{2} \int_0^2 (4 - t^2) \cos \left(\frac{i\pi t}{2} \right) dt = -t^2 \frac{2}{i\pi} \sin \left(\frac{i\pi t}{2} \right) \Big|_0^2 + \frac{2}{i\pi} \int_0^2 2t \sin \left(\frac{i\pi t}{2} \right) dt$$

$$= -\frac{4}{(i\pi)^2} \int_0^2 2td \cos\left(\frac{i\pi t}{2}\right) = -\frac{8}{(i\pi)^2} \left[t \cos\left(\frac{i\pi t}{2}\right) \Big|_0^2 + \int_0^2 \cos\left(\frac{i\pi t}{2}\right) dt \right] = (-1)^{i+1} \frac{16}{(i\pi)^2}$$

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2} \left(\cos \frac{\pi t}{2} - \frac{1}{2^2} \cos \frac{2\pi t}{2} + \frac{1}{3^2} \cos \frac{3\pi t}{2} \dots \right)$$

$$\text{Let } t=0, \text{ then } 4 = \frac{8}{3} + \frac{16}{\pi^2} \left(1 - \frac{1}{2^2} + \frac{1}{3^2} \dots \right) \text{ or } \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$\text{Let } t=2, \text{ then } 0 = \frac{8}{3} + \frac{16}{\pi^2} \left(\cos \pi - \frac{1}{2^2} \cos 2\pi + \frac{1}{3^2} \cos 3\pi - \frac{1}{4^2} \cos 4\pi \dots \right)$$

$$0 = \frac{8}{3} + \frac{16}{\pi^2} \left(-1 - \frac{1}{2^2} - \frac{1}{3^2} \cos 3\pi - \frac{1}{4^2} \right)$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{\pi^2}{4} = 2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) \text{ or } 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

§ Half-Range expansion

Theorem. (Half-range sine expansion)

If $f(t)$, originally, defined for $0 \leq t \leq p$, is extended to the interval

$-p \leq t \leq p$ as an odd function $f(-t) = -f(t)$ for $-p \leq t \leq p$. Then the odd

function $g(t)$ is given by

$$g(t) = \begin{cases} -f(-t) & -p \leq t \leq 0 \\ f(t) & 0 \leq t \leq p \end{cases}$$

then for the half-range sine expansion of the extended function,

$$f(t) = \sum_{i=1}^{\infty} b_i \sin \frac{i\pi t}{p}, \quad -p \leq t \leq p$$

where

$$b_i = \frac{2}{p} \int_0^p f(t) \sin \frac{i\pi t}{p} dt$$

Theorem. (Half-range cosine expansion)

If $f(t)$, originally, defined for $0 \leq t \leq p$, is extended to the interval

$-p \leq t \leq p$ as an even function $f(-t) = f(t)$ for $-p \leq t \leq p$. Then the even

function $g(t)$ is given by

$$g(t) = \begin{cases} f(-t) & -p \leq t \leq 0 \\ f(t) & 0 \leq t \leq p \end{cases}$$

then for the half-range sine expansion of the extended function,

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_i \cos \frac{i\pi t}{p}, \quad -p \leq t \leq p$$

where

$$a_0 = \frac{1}{p} \int_0^p f(t) dt, \quad a_i = \frac{2}{p} \int_0^p f(t) \cos \frac{i\pi t}{p} dt$$

Ex. Find the sine and cosine expansions of $f(t) = t$ for $0 \leq t \leq \pi$

a. Sine $f(t) = \sum_{i=1}^{\infty} b_i \sin it, \quad -\pi \leq t \leq \pi$

where

$$b_i = \frac{2}{\pi} \int_0^{\pi} t \sin(it) dt = -\frac{2}{i\pi} \int_0^{\pi} t d \cos(it) = -\frac{2}{i\pi} [t \cos(it)]_0^{\pi} - \int_0^{\pi} \cos(it) dt = \frac{2}{\pi} (-1)^{i+1}$$

b. Cosine $f(t) = a_0 + \sum_{i=1}^{\infty} a_i \cos(it)$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} t dt = \frac{\pi}{2},$$

$$a_i = \frac{2}{\pi} \int_0^{\pi} t \cos(it) dt = \frac{2}{\pi} \frac{1}{i} [t \sin(it)]_0^{\pi} - \int_0^{\pi} \sin(it) dt = \frac{2}{\pi} \frac{1}{i^2} \cos(it) \Big|_0^{\pi} = -\frac{2}{i^2 \pi} [1 - (-1)^i]$$

Hw. Find the sine expansion and cosine expansion of the function

$$f(t) = \begin{cases} \sin t & 0 < t \leq \pi/2 \\ 0 & \pi/2 < t \leq \pi \end{cases}$$

Hw.

(a) what is the half-range sine expansion of the function $f(t) = t^2$,

$$0 \leq t \leq 1$$

(b) Show that $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$

© Using the integration of (a) to show that $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{\pi^4}{96}$

§ Complex form

$$f(t) = a_0 + \sum_{i=1}^n [a_i \cos(\frac{i\pi t}{p}) + b_i \sin(\frac{i\pi t}{p})]$$

$$\cos(\frac{i\pi t}{p}) = \frac{1}{2} [e^{j \frac{i\pi t}{p}} + e^{-j \frac{i\pi t}{p}}], \quad \sin(\frac{i\pi t}{p}) = \frac{1}{2j} [e^{j \frac{i\pi t}{p}} - e^{-j \frac{i\pi t}{p}}]$$

$$\begin{aligned}
f(t) &= a_0 + \sum_{i=1}^n \left\{ a_i \frac{1}{2} [e^{j\frac{i\pi t}{p}} + e^{-j\frac{i\pi t}{p}}] + b_i \frac{1}{2j} [e^{j\frac{i\pi t}{p}} - e^{-j\frac{i\pi t}{p}}] \right\} \\
&= a_0 + \sum_{i=1}^n \left\{ \frac{1}{2} (a_i - jb_i) e^{j\frac{i\pi t}{p}} + \frac{1}{2} (a_i + jb_i) e^{-j\frac{i\pi t}{p}} \right\} \equiv c_0 + \sum_{i=1}^n \{ c_i e^{j\frac{i\pi t}{p}} + c_{-i} e^{-j\frac{i\pi t}{p}} \} \\
&= + \sum_{i=-n}^n c_i e^{j\frac{i\pi t}{p}}
\end{aligned}$$

where

$$c_i = \frac{1}{2} (a_i - jb_i) = \frac{1}{2p} \int_{-p}^p f(t) [\cos(\frac{i\pi t}{p}) - j \sin(\frac{i\pi t}{p})] dt = \frac{1}{2p} \int_{-p}^p f(t) e^{-j\frac{i\pi t}{p}} dt$$

$$c_{-i} = \frac{1}{2p} (a_i + jb_i) = \frac{1}{2p} \int_{-p}^p f(t) e^{j\frac{i\pi t}{p}} dt$$

$$a_0 = \frac{1}{2p} \int_{-p}^p f(t) dt = \frac{1}{2p} \int_{-p}^p f(t) e^{j\frac{0\pi t}{p}} dt = c_0$$

Ex. Find the complex Fourier series representation of

$$f(t) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 4 \end{cases}$$

$$c_0 = \frac{1}{4} \int_0^4 f(t) dt = \frac{1}{4} \int_1^4 1 dt = \frac{3}{4}, \quad c_i = \frac{1}{4} \int_1^4 e^{-j\frac{i\pi t}{2}} dt = \frac{j}{2i\pi} e^{-j\frac{i\pi t}{2}} \Big|_1^4 = \frac{j}{2i\pi} (1 - e^{-j\frac{i\pi}{2}})$$

Hw.

(a) Find the complex form of the Fourier series of the function

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases}$$

(b) Find the particular solution of $y'' + 3y' + 2y = f(t)$